AN EXTENSION OF THE GUM TREE METHOD FOR COMPLEX NUMBERS

B. D. Hall
Measurement Standards Laboratory of New Zealand,
Industrial Research Ltd, PO Box 31-310, Lower Hutt, New Zealand.

Abstract

An uncertain complex number is an entity that encapsulates information about an estimate of a complex quantity. Simple algorithms can propagate uncertain complex numbers through an arbitrary sequence of data-processing steps, providing a flexible tool for uncertainty calculations supported by software. The technique has important applications in modern instrumentation systems. Some radio-frequency measurements are discussed.

1 Introduction

Some difficulty has been reported [1] in evaluating the measurement uncertainty of instrumentation systems using the recommendations of the *Guide to the Expression of Uncertainty in Measurement* (GUM) [2]. The operation of modern systems is often thought of in terms of more than one step of data-processing, instead of the single measurement function required by the GUM's *Law of Propagation of Uncertainty* (LPU). Moreover, a complete sequence of steps may not be carried out at each measurement (e.g., instrument calibration contributes to uncertainty but is not usually repeated each time). Also, if a function of measurement results is of interest, the handling of uncertainties associated with systematic errors requires special care.

For measurements of real-valued quantities, a *GUM Tree* design technique for modular systems has been developed to address these issues [3]. It introduces the notion of an *uncertain number*, an entity encapsulating information about an estimate of a quantity that can be propagated using simple algorithms [4, 5]. GUM Tree systems provide full LPU uncertainty propagation for multi-step data processing, which can be represented succinctly using an uncertain number notation (much like matrix and complex number notations represent lower-level computations).

This paper introduces the idea of an *uncertain complex number* (UC) for complex quantities based on a bivariate form of the LPU that supports multi-step data-processing [6]. The essential features of the GUM Tree technique are unchanged: the algorithms and structure of an UC are analogous to the real-valued counterparts, although matrices and complex numbers replace real-number parameters.

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2 Mathematical Details

A one-step approach to propagation of uncertainty in complex-valued problems has been described [7]. This section presents a mathematically equivalent procedure that can be extended to multi-step problems.

In the notation used, complex quantities are in bold italics, like $\boldsymbol{t}$ or $\boldsymbol{\alpha}$; bold capitalized Roman letters represent matrices, like $\mathbf{R}$. Real quantities are in plane italics and the real and imaginary components of a complex number are labeled 1 and 2, respectively, like $\boldsymbol{t} = t_1 + \mathrm{i}t_2$. The standard uncertainty of a scalar t_1 is written $u(t_1)$ and the correlation associated with scalars t_1 and t_2 is written $r(t_1, t_2)$.

2.1 Single-Step Uncertainty Evaluation

A measurement procedure for a complex quantity of interest, the *measurand*, can be represented as a function of all the quantities that influence the measurement,

$$\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_l) \; . \tag{1}$$

An estimate $\boldsymbol{y}$ can be obtained by evaluating this function with influence quantity estimates, $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_l$. A standard uncertainty, $u(x_{ij})$, $i = 1, 2;\ j = 1, \cdots, l$, is associated with each of the real and imaginary components.

A measure of the contribution to uncertainty in $\boldsymbol{y}$ from the estimate $\boldsymbol{x}_j$ is the *component of uncertainty* matrix

$$\mathbf{U}_{\boldsymbol{x}_j}(\boldsymbol{y}) = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}_j}\, \mathbf{U}_{x_j}(\boldsymbol{x}_j) \tag{2}$$

$$= \begin{bmatrix} \frac{\partial y_1}{\partial x_{1j}} & \frac{\partial y_1}{\partial x_{2j}} \\ \frac{\partial y_2}{\partial x_{1j}} & \frac{\partial y_2}{\partial x_{2j}} \end{bmatrix} \begin{bmatrix} u(x_{1j}) & 0 \\ 0 & u(x_{2j}) \end{bmatrix} \; . \tag{3}$$

The component of uncertainty matrices associated with the influence quantities can be used to evaluate the variance-covariance matrix associated with $\boldsymbol{y}$, which characterises the measurement uncertainty,

$$\mathbf{V}(\boldsymbol{y}) = \sum_{i=1}^{l} \sum_{j=1}^{l} \mathbf{U}_{\boldsymbol{x}_i}(\boldsymbol{y})\, \mathbf{R}(\boldsymbol{x}_i, \boldsymbol{x}_j)\, \mathbf{U}'_{\boldsymbol{x}_j}(\boldsymbol{y}) \tag{4}$$

$$= \begin{bmatrix} u^2(y_1) & u(y_1, y_2) \\ u(y_1, y_2) & u^2(y_2) \end{bmatrix} \; , \tag{5}$$

where the correlation coefficients associated with a pair of complex quantities, or of a single quantity, are the elements of[1]

$$\mathbf{R}(\boldsymbol{x}_i, \boldsymbol{x}_j) = \begin{bmatrix} r(x_{1i}, x_{1j}) & r(x_{1i}, x_{2j}) \\ r(x_{2i}, x_{1j}) & r(x_{2i}, x_{2j}) \end{bmatrix} \; . \tag{6}$$

[1] Note, this matrix is not symmetric, because, in general, $r(x_{1i}, x_{2j}) \neq r(x_{2i}, x_{1j})$.

2.2 Multi-Step Uncertainty Problems

Some procedures are better represented by a sequence of equations[2]

$$\boldsymbol{Y}_1 = \boldsymbol{f}_1(\Lambda_1) \; , \; \cdots \; , \; \boldsymbol{Y}_p = \boldsymbol{f}_p(\Lambda_p) \; , \tag{7}$$

where Λ_i is the set of arguments of the i^{th} function,[3] and $\boldsymbol{Y} = \boldsymbol{Y}_p$.

The uncertainty calculation (4) still applies. However, each component of uncertainty in $\boldsymbol{y}$ is now evaluated recursively; the i^{th} step being[4]

$$\mathbf{U}_{\boldsymbol{x}_j}(\boldsymbol{f}_i) = \sum_{\boldsymbol{y}_k \in \Lambda_i} \frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{y}_k} \mathbf{U}_{\boldsymbol{x}_j}(\boldsymbol{y}_k) \; . \tag{8}$$

This is a key result. It shows that all that is required to obtain the components of uncertainty of an intermediate result $\boldsymbol{y}_i$ are the partial derivatives of $\boldsymbol{f}_i$ with respect to its direct inputs (elements of Λ_i) and the components of uncertainty associated with those inputs.[5] The result shows that data processing can be arranged to evaluate the result and the components of uncertainty at an intermediate step before proceeding to the next step. This is exploited by the GUM Tree technique to automate the step-by-step evaluation of intermediate results and components of uncertainty in software.

3 Uncertain Complex Numbers

The notion of an UC entity arises from the concurrent processing of intermediate results and components of uncertainty in multi-step problems. By associating UCs with quantity estimates, data processing can be expressed succinctly in terms that are directly related to the physical quantities involved, while the low-level propagation of uncertainty is hidden. Moreover, UC expressions can be programmed into a computer without additional analysis, such as derivation of sensitivity coefficients, etc [3].[6]

Some terminology is associated with UC notation. For example, a UC equation corresponding to (1) could be written as $\mathtt{y} = \mathtt{f}(\mathtt{x}_1, \cdots, \mathtt{x}_l)$, with typewriter font used to represent UCs. The function arguments are all associated with influence quantity estimates: a complex value and a pair of standard uncertainties define these UCs, which are referred to as *elementary*. On the other hand, `y` is an *intermediate* UC, defined by a function.[7] UCs have a number of numerical properties that can be evaluated, including *value* and *component of uncertainty*. Implicitly, the UC function `f()` defines $\boldsymbol{y}$ as well as the set of component of uncertainty matrices required to obtain the covariance in (4). The value of `y` is the estimate $\boldsymbol{y}$, obtained by processing data according to the

[2] For the description of multi-step calculations, we now adopt the convention that *only* influence quantities are represented by $\boldsymbol{X}$, or $\boldsymbol{x}$, whereas $\boldsymbol{Y}$, or $\boldsymbol{y}$, may be used for *either* influences *or* (intermediate) results. The measurand may be written without a subscript, but all other intermediate quantities are subscripted.

[3] These arguments may be the results of earlier steps, or influence quantities.

[4] Recursion stops when $\boldsymbol{y}_k$ refers to an influence estimate, not an intermediate result.

[5] In this equation, $\frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{y}_k}$ is actually the Jacobian matrix of the bivariate function (c.f. equation (3)), so the method applies even when the complex function $\boldsymbol{f}_i$ is not analytic.

[6] Software support provides an UC data type and a library of basic mathematical operations that can be used to form arbitrary UC expressions.

[7] The distinction between elementary and intermediate UCs is analogous to the independent and dependent variables of ordinary mathematics.

defining equations in the usual way. The component of uncertainty of $\mathtt{y}$ with respect to $\mathtt{x_j}$ is the matrix $\mathbf{U}_{\boldsymbol{x}_j}(\boldsymbol{y})$ defined in (3). In software, $\mathtt{f}()$ is automatically decomposed into a sequence of basic UC operations that evaluate $\mathtt{y}$ in a procedure analogous to the evaluation of $\boldsymbol{y}$ by basic complex operations [5].

4 Application and Discussion

The application of UCs to some important types of radio-frequency (RF) measurement has been investigated recently: measurement of RF power, and measurement of complex reflection coefficients with a vector network analyser (VNA) were studied [8]. A common feature of these studies was the importance of the uncertainty arising from instrument 'calibration'.[8] A calibration procedure estimates measurement errors that can be adjusted during normal instrument operation. This is generally done by recording the instrument's response to a number of well-characterised reference stimuli.

A calibration procedure will give rise to a number of equations of the form[9]

$$\mathtt{f}_{\mathrm{cal}}(\mathtt{a}_i, \mathtt{b}_i, \cdots, \mathtt{v}, \mathtt{w}, \cdots; \mathtt{E}_1, \mathtt{E}_2, \cdots; \boldsymbol{\alpha}_i, \boldsymbol{\gamma}_i) = 0 \;, \tag{9}$$

which are solved to obtain the correction parameters $\mathtt{E}_1, \mathtt{E}_2, \cdots$. The UCs $\mathtt{a}_i, \mathtt{b}_i, \cdots, \mathtt{v}, \mathtt{w}, \cdots$ are associated with errors: indexed terms ($\mathtt{a}_i, \mathtt{b}_i$, etc), refer to independent errors (e.g., electronic instrumentation noise), while $\mathtt{v}, \mathtt{w}, \cdots$ are associated with fixed errors (e.g., the unknown error in a particular reference value). The nominal value of the reference, $\boldsymbol{\alpha}_i$, and the instrument's response, $\boldsymbol{\gamma}_i$, are pure numbers with no uncertainty.

During normal operation, an adjustment function is used to map an instrument response to an UC for the measured quantity. This can be represented by an equation of the form[9]

$$\mathtt{y}_i = \mathtt{f}_{\mathrm{adj}}(\mathtt{a}_i, \mathtt{b}_i, \cdots, \mathtt{v}, \mathtt{w}, \cdots; \mathtt{E}_1, \mathtt{E}_2, \cdots; \boldsymbol{\gamma}_i) \;, \tag{10}$$

where $\mathtt{a}_i, \mathtt{b}_i, \cdots, \mathtt{v}, \mathtt{w}, \cdots$ are the independent and systematic uncertainties and $\boldsymbol{\gamma}_i$, is the uncorrected instrument response, with no uncertainty. The correction parameters $\mathtt{E}_1, \mathtt{E}_2, \cdots$ here ensure that uncertainties due to calibration errors are propagated correctly to the measurement result $\mathtt{y}_i$.

The RF calibration and measurement procedures were described in these studies as sequences of equations. Using UC data processing software, the correction parameter uncertainties from calibration were propagated in the adjustment functions and measurement results were processed further, to obtain UCs for the ratio and difference of measurements. At all stages, intermediate UC properties could be evaluated, allowing an assessment to made of the relative importance of various errors. The results showed that errors during calibration introduced correlation between measurement results, which, when propagated further, affected the uncertainty of the ratios and differences significantly. For instance, in the VNA measurement of reflection coefficients, the uncertainty associated with the calculated ratios and differences was a factor of 10 lower than a similar calculation that treated the measurement results as independent.[10] Similarly, in the case of RF power measurements, the relative uncertainty of the

[8] Calibration is colloquial; 'correction' is a more appropriate metrological term.

[9] Although we write this as a single equation, it is often a sequence of equations.

[10] The trace of the covariance matrices was used as the magnitude of the uncertainty.

power measurement ratios was actually less than the relative uncertainty of the individual measurements, because of uncertainty about calibration error in the meter's scale factor. Had the measurement uncertainties been treated as independent, the relative uncertainty in the ratio would have been more than double.

It is no surprise that calibration errors lead to correlation between measurement results. However, a comprehensive uncertainty analysis of these measurement scenarios is rarely undertaken because of the complexity of the error models. Our study shows that an UC analysis is straightforward when problems are formulated as a sequence of equations.

The GUM Tree technique, and the associated idea of an uncertain complex number, are new tools to apply established methods of uncertainty analysis. They provide a powerful algorithmic approach that is particularly suited to large, complicated problems, such as those presented by many modern measuring instruments. They also provide a way of seamlessly propagating uncertainty during any post-processing of measurement results. Indeed, now that embedded computers are an ubiquitous part of most measuring instruments, there is ample opportunity for designers to incorporate these tools as internal routines. To do so would herald a new generation of measurement systems: capable of reporting both an estimate of the *value* of a quantity of interest and most, if not all, of the information needed to evaluate the *uncertainty* in that estimate.

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