Computing uncertainty with Uncertain Numbers

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Abstract

A computational method for processing measurement data is discussed that applies the 'Law of Propagation of Uncertainty', which is described in the *Guide to the Expression of Uncertainty in Measurement*. The method introduces the notion of an 'uncertain number', which is an entity that encapsulates information about the value and uncertainty of a quantity. Basic mathematical operations can be defined for uncertain numbers so that relationships between quantities can be expressed in a familiar way. Uncertain number equations can be arbitrarily combined, or decomposed into expressions for intermediate results, providing flexibility and convenience for automatic data processing while rigorously supporting uncertainty propagation. A geometrical description of simple uncertain-number operations is also given.

1 Introduction

This article introduces the notion of an *uncertain number* as an entity that encapsulates information about the value and uncertainty of a quantity. Uncertain numbers are useful as an abstract representation of measurement data subject to uncertainty.

Uncertain number calculations apply the so-called 'Law of Propagation of Uncertainty' (LPU), described in the *Guide to the Expression of Uncertainty in Measurement* (*Guide*) [1, 5.1.2].[1] However, the computational method is complementary to 'direct' applications of the LPU; i.e., where a complete, closed-form, expression of a measurement function is known and sensitivity-coefficient expressions have to be derived from it for all influence quantities [1]. One of the current difficulties in following *Guide* recommendations is that the complexity of many measurement problems makes strict formulation of a direct LPU calculation impractical. Approximations can be used to simplify calculations,[2] but require specialist skill to develop and validate. A similar issue underlies the difficulty of supporting uncertainty calculations in instrumentation systems [2]. How can the behavior of complicated systems be captured in a single measurement equation suitable for LPU analysis? How should modular systems be handled, where sensors or subsystems may be changed at any time, thereby altering the measurement function? Indeed, it seems paradoxical that modern instruments and instrumentation systems

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[1]The method of calculation uses a linear approximation for the measurement function in the vicinity of the input parameter values. In cases where non-linearity is significant on the scale of the measurement uncertainties involved, more careful consideration of the calculation method may be required.

[2]For instance, it is sometimes possible to anticipate the near cancellation of particular influence terms at some stage of a calculation, which allow them to be dropped altogether.

perform quite elaborate data processing, but provide no support for uncertainty calculation.

The computational procedure associated with uncertain numbers is well-suited to data processing, which generally involves a series of intermediate steps that are chained together.[3] The rules governing uncertain number calculations incorporate both the chain rule for function composition and the chain rule for partial differentiation. This allows value and uncertainty information to be propagated simultaneously through any number of computational steps and is analogous to the flow of data in a modular instrumentation system. In this way, the uncertain number approach tackles the issue of complexity by allowing computations to be handled in appropriate intermediate steps. This inherent support for modularity, and the ease of automating uncertain number calculations, could enable greater uptake of the recommendations of the *Guide*.

Uncertain-number calculations will generally be longer 'by hand' than direct LPU calculations. However, the procedure is easily automated. With full software support a calculation can be defined simply by specifying a set of measurement equations. There is no need to supply additional expressions for the sensitivity of a result to influence quantities, because these can be obtained algorithmically. In this way, uncertain numbers can be thought of as abstract entities representing the inherent variability of observations, much like random variables [1, Section 4.1]. An uncertain number's value and uncertainty correspond, respectively, to the mean and standard deviation of a random variable.

This article describes the properties of uncertain numbers and the rules that govern calculations with them. An example shows how data processing can be conveniently handled in intermediate steps using the new approach. A geometrical interpretation of simple mathematical operations is also provided.

2 Uncertain numbers

The next two sections describe the computational details governing uncertain numbers. As much as possible, terminology is aligned with usage found in the *Guide*, although some particular terms relating to uncertain numbers are introduced. Section 2.3 illustrates the computational procedure with an example. Then, in section 2.4, a geometrical interpretation is given.

2.1 Terminology and properties of uncertain numbers

It is helpful to make a distinction between two categories of uncertain number. The term *elementary* applies to an uncertain number that is directly assigned a value, a standard uncertainty and a number of degrees of freedom, as opposed to an uncertain number that is defined by an equation of uncertain numbers. Uncertain numbers will be shown in boldface Roman type below. The symbol $\mathbf{x}$ is reserved exclusively for elementary uncertain numbers; $\mathbf{y}$ is used when an uncertain number need not be elementary.

Every uncertain number $\mathbf{y}$ has an associated influence set, $\mathcal{I}_y$, identifying the elementary uncertain numbers that influence $\mathbf{y}$. The elements of $\mathcal{I}_y$ are written in italics, e.g., $x_i \in \{x_1, x_2, \cdots\}$; if $\mathbf{y}$ is elementary, $\mathcal{I}_y = \{y\}$.

[3] For example, a sequence of keystroke operations on a pocket calculator.

There are a number of properties that can be calculated for $\mathbf{y}$.

- A *value*, y.[4] If $\mathbf{y}$ is elementary, the value is assigned. If not, the value is calculated, as described in section 2.2.
- A *component of uncertainty*, $u_x(y)$, representing the change that would occur in y if the value of the influence quantity, x, were to change by an amount equal to its standard uncertainty.[5] If $\mathbf{y}$ is elementary, $u_y(y)$ is equal to the standard uncertainty $u(y)$. If not, the component of uncertainty is calculated, as described in section 2.2.
- A *standard uncertainty*,

$$u(y) = \left[\sum_{x_i, x_j \in \mathcal{I}_y} u_{x_i}(y)\, r(x_i, x_j)\, u_{x_j}(y) \right]^{1/2} , \tag{1}$$

  where $r(x_i, x_j)$ represents a correlation coefficient that may be assigned between any pair of elementary uncertain numbers $\mathbf{x}_i$ and $\mathbf{x}_j$.[6] This is equivalent to the "combined standard uncertainty" defined by equation (13) in Section 5.2.2 of the *Guide*. If $\mathbf{y}$ is elementary, the standard uncertainty has been assigned.

- a number of *degrees of freedom*, ν_y, which can be found from

$$\frac{u^4(y)}{\nu_y} = \sum_{x_i \in \mathcal{I}_y} \frac{u^4_{x_i}(y)}{\nu_{x_i}} . \tag{2}$$

  Note that this calculation does not apply when there is correlation between the elementary uncertain numbers that influence $\mathbf{y}$. In such cases the number of degrees of freedom is not defined. This equation is equivalent to *Guide* equation (G.2a). If $\mathbf{y}$ is elementary, the number of degrees of freedom has been assigned.

2.2 Functions of uncertain numbers

Suppose that an uncertain number is defined by the equation

$$\mathbf{y} = f(\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p) , \tag{3}$$

where $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p$ need not be elementary. The influence set of $\mathbf{y}$ is the union of the influence sets

$$\mathcal{I}_y = \mathcal{I}_{y1} \bigcup \mathcal{I}_{y2} \bigcup \cdots \bigcup \mathcal{I}_{yp} \tag{4}$$

and the value of $\mathbf{y}$ is simply

$$y = f(y_1, y_2, \cdots, y_p) . \tag{5}$$

[4] Note, the same italic notation is used for the 'value' of an uncertain number and for set-elements that refer to uncertain numbers. The intended meaning should be clear from the context.

[5] The component of uncertainty is a signed real number. In the *Guide*, the closely related definition of a 'component of combined standard uncertainty' [1, Annex J] is equal to its absolute value.

[6] Note that $r(x_i, x_i) \equiv 1$ and, when $i \neq j$, it is assumed that $r(x_i, x_j) = 0$ unless an assignment is made.

$y = f(y_1, \cdots)$	$u_x(y) = \frac{\partial y}{\partial y_1} u_x(y_1) + \cdots$
$y = a y_1$	$u_x(y) = a\, u_x(y_1)$
$y = y_1^n$	$u_x(y) = n\, \frac{y}{y_1}\, u_x(y_1)$
$y = \sin(y_1)$	$u_x(y) = \cos(y_1)\, u_x(y_1)$
$y = \cos(y_1)$	$u_x(y) = -\sin(y_1)\, u_x(y_1)$
$y = y_1 + y_2$	$u_x(y) = u_x(y_1) + u_x(y_2)$
$y = y_1 - y_2$	$u_x(y) = u_x(y_1) - u_x(y_2)$
$y = y_1\, y_2$	$u_x(y) = \frac{y}{y_1}\, u_x(y_1) + \frac{y}{y_2}\, u_x(y_2)$
$y = y_1 / y_2$	$u_x(y) = \frac{y}{y_1}\, u_x(y_1) - \frac{y}{y_2}\, u_x(y_2)$

Table 1: Some simple mathematical operations and their corresponding component of uncertainty expressions in terms of an unspecified influence quantity x (see equation (6)). In general, the arguments, $y_1, \cdots$, represent values obtained at intermediate steps of data processing.

The component of uncertainty, due to the uncertainty of any elementary $\mathbf{x}$, is

$$u_x(y) = \sum_{y_k \in \{y_1, y_2, \cdots, y_p\}} \frac{\partial y}{\partial y_k}\, u_x(y_k)\ . \qquad (6)$$

Equations (4), (5) and (6) are recursive, the calculation of these equations stops when elementary uncertain numbers are encountered. The practical advantages of uncertain numbers, over 'direct' evaluations of the LPU, stem from the recursive nature of the computations.

Most equations that arise in measurement can be evaluated as a sequence of basic operations. Common univariate and bivariate functions have quite simple expressions for $u_x(y)$, some of which are shown in Table 1. So, the uncertain numbers approach can be implemented using only a small number of predetermined component-of-uncertainty expressions: no explicit use of differentiation is required beyond this.

2.3 Example

To see how uncertain number calculations work, a scenario will be helpful. The following example is presented in a way that may seem inappropriate to readers familiar with direct LPU calculations, because it begins by considering the last two data-processing steps that obtain the results required. However, this part of the calculation can be defined, in terms of uncertain numbers, before there is full knowledge about the quantities that contribute to the measurement uncertainty. The example then shows how the remaining details of a calculation are resolved when a specific measurement uncertainty model is chosen. Finally, the example shows that parts of the measurement model can be changed without difficulty.

Suppose that the difference in electrical potential across a resistance has been measured and that the current flowing circuit and the power dissipated are to be calculated. We begin by associating uncertain numbers with the measured quantities: $\mathbf{V}$, for the electrical potential and $\mathbf{R}$, for the resistance. Next, we *define* uncertain numbers for the current and power in terms of voltage and resistance

$$\mathbf{I} = \mathbf{V}/\mathbf{R}$$

and

$$\mathbf{P} = \mathbf{VI} \, .$$

For the current, **I**, this implies that the influence set, value, and component of uncertainty are, respectively (see Table 1),

$$\begin{aligned} \mathcal{I}_I &= \mathcal{I}_V \bigcup \mathcal{I}_R && (7) \\ I &= V/R && (8) \\ u_x(I) &= \frac{I}{V} u_x(V) - \frac{I}{R} u_x(R) \, , && (9) \end{aligned}$$

where x refers to an unspecified elementary uncertain number.[7] Similarly, for the power, **P**, the influence set, value and component of uncertainty are, respectively (see Table 1),

$$\begin{aligned} \mathcal{I}_P &= \mathcal{I}_V \bigcup \mathcal{I}_I && (10) \\ P &= VI \, , && (11) \\ u_x(P) &= \frac{P}{V} u_x(V) + \frac{P}{I} u_x(I) \, . && (12) \end{aligned}$$

In a sense, this completes the analysis of the uncertain-number data processing required to obtain **I** and **P**. Equations (7) to (12) are quite general, they apply equally well to measurement scenarios where **V** and **R** are elementary, or to measurements where **V** and **R** have other influences.

2.3.1 Elementary V and R

In the simplest case, **V** and **R** can be defined as elementary uncertain numbers if numerical values are available for the potential difference, V, the resistance, R, and the standard uncertainties, $u(V)$ and $u(R)$.[8]

With a complete model for the influence quantities in the measurement, it is possible to see how the general expressions (7), (9), (10) and (12) apply. From (7), the influence set $\mathcal{I}_I$ is $\{V, R\}$, the union of $\{V\}$ and $\{R\}$. So, there are two expressions for the components of uncertainty in **I**: $u_V(I)$, due to uncertainty in V, and $u_R(I)$, due to uncertainty in R

$$\begin{aligned} u_V(I) &= \frac{I}{V} u(V) + 0 \, , && (13) \\ u_R(I) &= 0 - \frac{I}{R} u(R) \, , && (14) \end{aligned}$$

using $I = V/R$ and the fact that $u_R(V) = 0$ and $u_V(R) = 0$. The influence set for **P** is $\{V, R\}$, the union of $\{V\}$ and $\{V, R\}$. So, according to (12), the component of

[7] It is conceivable that, depending on the model of uncertainty chosen for these measurements, **V** and **R** will be functions of other uncertain numbers.

[8] Degrees of freedom are not considered in this example.

uncertainty expressions are

$$u_V(P) = \frac{P}{V}\,u(V) + \frac{P}{I}\,u_V(I)\ , \tag{15}$$

$$u_R(P) = 0 + \frac{P}{I}\,u_R(I)\ , \tag{16}$$

where $P = VI$, $u_V(I)$ and $u_R(I)$ refer to (13) and (14), respectively, and $u_R(V) = 0$.

2.3.2 A temperature dependent resistance

Now suppose that a more detailed model of the resistance accounts for temperature sensitivity. For example,

$$R = R_0\,[1 + \alpha(T - T_0)]\ , \tag{17}$$

where R_0 is the resistance at a temperature T_0, α is a temperature coefficient and T is the temperature of the resistor during the measurement. Assume that R_0, T_0, T and α are all subject to uncertainty. We can associate uncertain numbers with these quantities and, although not strictly necessary, evaluate equation (17) in a series of basic operations to mimic a simple computer-based calculation,

$$\begin{aligned} \mathbf{\Delta T} &= \mathbf{T} - \mathbf{T}_0\ , \\ \mathbf{\Delta R} &= \boldsymbol{\alpha}\mathbf{R}_0\mathbf{\Delta T}\ , \\ \mathbf{R} &= \mathbf{R}_0 + \mathbf{\Delta R}\ . \end{aligned}$$

Allowing for the possibility that $\mathbf{R}_0$, $\mathbf{T}_0$, $\mathbf{T}$ and $\boldsymbol{\alpha}$ are not elementary, [9] the influence sets for $\mathbf{\Delta T}$, $\mathbf{\Delta R}$ and $\mathbf{R}$ are

$$\mathcal{I}_{\Delta T} = \mathcal{I}_T \bigcup \mathcal{I}_{T_0}\ , \tag{18}$$

$$\mathcal{I}_{\Delta R} = \mathcal{I}_\alpha \bigcup \mathcal{I}_{R_0} \bigcup \mathcal{I}_{\Delta T}\ , \tag{19}$$

$$\mathcal{I}_R = \mathcal{I}_{R_0} \bigcup \mathcal{I}_{\Delta R}\ . \tag{20}$$

and the corresponding component of uncertainty expressions are (from Table 1), for any influence quantity x,

$$u_x(\Delta T) = u_x(T) - u_x(T_0)\ , \tag{21}$$

$$u_x(\Delta R) = \frac{\Delta R}{\alpha}\,u_x(\alpha) + \frac{\Delta R}{R_0}\,u_x(R_0) + \frac{\Delta R}{\Delta T}\,u_x(\Delta T)\ , \tag{22}$$

$$u_x(R) = u_x(R_0) + u_x(\Delta R)\ . \tag{23}$$

When numerical values for R_0, $u(R_0)$, T_0, $u(T_0)$, T, $u(T)$, α and $u(\alpha)$ are available, $\mathbf{R}_0$, $\mathbf{T}_0$, $\mathbf{T}$ and $\boldsymbol{\alpha}$ will be elementary uncertain numbers. In that case, the component-of-uncertainty calculations for $\mathbf{I}$ and $\mathbf{P}$ are as follows.

The influence set $\mathcal{I}_{\Delta T} = \{T_0, T\}$. So, equation (21) leads to two expressions for the

[9] For instance, there could be a detailed model of the uncertainty contributions associated with measuring temperature.

components of uncertainty[10]

$$\begin{aligned}
u_{T_0}(\Delta T) &= 0 - u(T_0)\ , && (24)\\
u_T(\Delta T) &= u(T) + 0\ , && (25)
\end{aligned}$$

using the fact that $u_T(T_0) = 0$ and $u_{t_0}(T) = 0$.

In the next step, $\mathcal{I}_{\Delta R} = \{R_0, T_0, T, \alpha\}$, so there are four components of uncertainty evaluated from (22)

$$\begin{aligned}
u_{R_0}(\Delta R) &= 0 + \frac{\Delta R}{R_0} u(R_0) + 0\ , && (26)\\
u_{T_0}(\Delta R) &= 0 + 0 + \frac{\Delta R}{\Delta T} u_{T_0}(\Delta T)\ , && (27)\\
u_T(\Delta R) &= 0 + 0 + \frac{\Delta R}{\Delta T} u_T(\Delta T)\ , && (28)\\
u_\alpha(\Delta R) &= \frac{\Delta R}{\alpha} u(\alpha) + 0 + 0\ , && (29)
\end{aligned}$$

where the values of $\Delta R = \alpha R_0 \Delta T$ and $\Delta T = T - T_0$ would be known, as well as the components of uncertainty $u_{T_0}(\Delta T)$ and $u_T(\Delta T)$. Terms such as $u_{T_0}(\alpha)$ and $u_T(R_0)$ are always zero.

In the step associated with R, $\mathcal{I}_R = \{R_0, T_0, T, \alpha\}$, so the components of uncertainty in the resistance, evaluated using (23), are

$$\begin{aligned}
u_{R_0}(R) &= u(R_0) + u_{R_0}(\Delta R)\ , && (30)\\
u_{T_0}(R) &= 0 + u_{T_0}(\Delta R)\ , && (31)\\
u_T(R) &= 0 + u_T(\Delta R)\ , && (32)\\
u_\alpha(R) &= 0 + u_\alpha(\Delta R)\ . && (33)
\end{aligned}$$

The influence set for current has five elements $\mathcal{I}_I = \{V, R_0, T_0, T, \alpha\}$. The components of uncertainty evaluated using (9) are therefore

$$\begin{aligned}
u_V(I) &= \frac{I}{V} u(V) + 0\ , && (34)\\
u_{R_0}(I) &= 0 - \frac{I}{R} u_{R_0}(R)\ , && (35)\\
u_{T_0}(I) &= 0 - \frac{I}{R} u_{T_0}(R)\ , && (36)\\
u_T(I) &= 0 - \frac{I}{R} u_T(R)\ , && (37)\\
u_\alpha(I) &= 0 - \frac{I}{R} u_\alpha(R)\ . && (38)
\end{aligned}$$

Similarly, the influence set $\mathcal{I}_P = \{V, R_0, T_0, T, \alpha\}$ and the five components of uncertainty

[10] Other components of uncertainty, for example, $u_\alpha(\Delta T)$ are zero, because any $u_x(y)$, where x is not a member of $\mathcal{I}_y$, must be zero.

evaluated by (12) are

$$
\begin{aligned}
u_V(P) &= \frac{P}{V}\,u(V) + \frac{P}{I}\,u_V(I)\ , & (39)\\
u_{R_0}(P) &= 0 + \frac{P}{I}\,u_{R_0}(I)\ , & (40)\\
u_{T_0}(P) &= 0 + \frac{P}{I}\,u_{T_0}(I)\ , & (41)\\
u_T(P) &= 0 + \frac{P}{I}\,u_T(I)\ , & (42)\\
u_\alpha(P) &= 0 + \frac{P}{I}\,u_\alpha(I)\ . & (43)
\end{aligned}
$$

In summary, the example first obtained expressions for the current and the power by attributing uncertain numbers to measurements of potential difference and resistance. These expressions were general. They allowed for the unspecified complexity of the underlying measurement model by referring to an arbitrary set of influence quantities. The example showed that when complete measurement uncertainty models are defined, the remaining details of the calculation can be determined. Lastly, it showed that uncertain number data processing can adapt, in a simple and predictable manner, to changes in the measurement uncertainty model used.

Throughout the example, basic mathematical operations were used to show that automation of the procedure is feasible and that differentiation of measurement equations may be avoided by restricting computational steps to a few tabulated functions. Of course, when the temperature-dependent resistance is introduced, the 3-step evaluation of (17) could be avoided by applying (6) directly.

2.4 Geometrical interpretation

Uncertain number calculations can be given a geometrical representation.

Let $\mathbf{y}$ be defined again by equation (3) with the influence set $\{x_1, x_2, \cdots, x_l\}$. The corresponding components of uncertainty may be written as a vector

$$\mathbf{U}_y = \begin{bmatrix} u_{x_1}(y) \\ u_{x_2}(y) \\ \vdots \\ u_{x_l}(y) \end{bmatrix} . \quad (44)$$

Further, let $\mathbf{R} = \{r_{ij}\}$ be the $l \times l$ matrix of correlation coefficients, with $r_{ij} = r(x_i, x_j)$. So, the calculation of standard uncertainty using equation (1) can be written as

$$u(y) = \left[\mathbf{U}'_y\,\mathbf{R}\,\mathbf{U}_y\right]^{1/2} . \quad (45)$$

When the elementary uncertain numbers that influence $\mathbf{y}$ are independent, $\mathbf{R} = \mathbf{I}_l$, the $l \times l$ identity matrix, and equation (45) becomes

$$u(y) = \left[\mathbf{U}'_y\,\mathbf{U}_y\right]^{1/2} . \quad (46)$$

This is just the magnitude of the vector $\mathbf{U}_y$. So, a system of mutually orthogonal axes can be associated with the l components of $\mathbf{U}_y$. The scale of these axes is the same as that of the measurand.

The component-of-uncertainty calculations associated with the addition and subtraction of uncertain numbers (see Table 1, for $\mathbf{y} = \mathbf{y}_1 \pm \mathbf{y}_2$) can now be seen to correspond to vector addition and subtraction ($\mathbf{U}_y = \mathbf{U}_{y_1} \pm \mathbf{U}_{y_2}$). Furthermore, the correlation that can arise when two derived quantities share influences (c.f., [1, F.1.2.3]) is neatly represented in this geometrical picture. If a pair of uncertain numbers $\mathbf{y}_1$ and $\mathbf{y}_2$ have common influences, their covariance is given by the scalar product

$$u(y_1, y_2) = \mathbf{U}'_{y_1}\,\mathbf{U}_{y_2}\;, \tag{47}$$

where the system of axes must include all the influences of $\mathbf{y}_1$ and $\mathbf{y}_2$.

More generally, some elementary uncertain numbers may be correlated. In such cases, the system of axes associated with $\mathbf{U}_y$ is not orthogonal. The cosine of the angle between the axes associated with x_i and x_j is given by the correlation coefficient $r(x_i, x_j)$ and $\mathbf{R}$ can be thought of as the metric tensor of the system. The scalar product of a pair of vectors $\mathbf{A}$ and $\mathbf{B}$ in such a system of axes is

$$\mathbf{A}'\,\mathbf{R}\,\mathbf{B}\,. \tag{48}$$

So, the calculation of standard uncertainty using equation (45) can still be interpreted as the magnitude of $\mathbf{U}_y$ and the other geometrical analogies mentioned continue to hold.

3 Discussion

A simple computational technique has been described that is suitable for processing measurement data. The mathematics is not complicated and no originality is claimed. However, the approach offers a view of the uncertainty calculation problem that is potentially powerful and could lead to greater uptake of uncertainty calculations compatible with the *Guide*. Uncertain numbers can be used to represent measured quantities, and sources of error, that are associated with random variables in the context of a measurement. This can simplify the analysis of complicated measurement problems. For example, an uncertain-number implementation of the full complex-valued error model of a microwave vector network analyser was undertaken recently [3, 4].[11]

One of the significant advantages of the uncertain numbers approach is the way that a large problem can be decomposed into distinct aspects and be worked on independently. For example, during the design of a measuring instrument, a measurement uncertainty model can be developed. Later, the instrument could use this model to provide uncertainty information in the form of uncertain numbers, thereby contributing to the overall uncertainty budget of the measurement procedure in which it is being used. Such modularity contrasts with the 'direct' approach to LPU calculation, which burdens instrument-users with the task of developing an uncertainty budget for the whole measurement procedure. This requires a detailed model of all contributions to measurement uncertainty to be formulated, validated and analysed mathematically to obtain a statement of uncertainty. With the uncertain numbers approach, there will

[11] The notion of complex-valued uncertain number is an extension of the real-valued case. Some of the properties of an uncertain complex number are multivariate. More details are available in [6].

be some parts of that task that can be carried out and validated independently. Most modern instrumentation already performs internal data processing in order to report a *value*. Why not provide information about the uncertainty, too?

It is one of the stated aims of the *Guide* that statements of uncertainty should be *transferable* [1, Section 0]. Uncertain numbers help to highlight what is needed to achieve this, because transferability is implicit in equations (4), (5) and (6). A close look at these equations allows a minimal communications interface for handling uncertainty information to be specified [5], which is an important consideration for measurement system design. The notion of an uncertain number therefore offers a high-level view of the correct way to handle data produced during intermediate stages of a measurement procedure.

Software support for uncertain numbers is desirable. By automating the computational process, uncertainty calculations can be applied to problems where the complexity, or lack of flexibility, of a direct LPU calculation is an obstacle. Software frameworks that support the systematic development, checking and validation of uncertainty calculations, can be developed on the basis of the simple rules that govern uncertain number calculations [7]. Uncertain number software tools are available for scalar and complex-valued measurement data processing [8].

In conclusion, this article describes a computational approach to the mathematics of the LPU that offers new scope for applying the recommendations of the *Guide*. The uncertain number is a convenient and intuitive notion, with a geometrical interpretation, that will be useful when measurement data processing calls for uncertainty propagation.

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